

Differential Equations (9.1, 9.3, 9.4) (intro to MATH 307)

What is a diff eq?

Mathematically, it's a functional equation that relates a function and its derivative(s).

The answer/solution to a differential equation is a function.

Ex.) $\frac{dy}{dx} = 4y$ (or $y' = 4y$)

function of x.

We're trying to find: $y(x)$ that makes the function equation true

1) Check that $y = e^{4x}$ is a solution to this diff eq.

$y = e^{4x}$
 $y' = 4e^{4x}$ } plug these in

$$4e^{4x} = 4 \cdot (e^{4x})$$

these are always equal

2) Is $y = x^3$ a solution to this differential equation? ($y' = 4y$)

$$y = x^3$$
$$y' = 3x^2$$

$$3x^2 = 4x^3$$

not the same function, so No.

3) Is $y = 0$ a solution to $y' = 4y$?

$$y' = 0$$

$$0 = 4 \cdot 0$$

always true so, Yes.

It turns out that the solutions to $y' = 4y$ look like: $y = Ae^{4t}$ for some constant A .

Ex) Verify that $y = e^{-t^2}$ is a solution to $y' = -2ty$.

independent variable function (dependent variable)

$$y = e^{-t^2}$$

$$y' = -2te^{-t^2}$$

$$-2te^{-t^2} = -2t(e^{-t^2})$$

always equal ✓

$y' = f(x)$, solution is an integral,
what we've been doing

"Philosophical" interpretation:
Differential Equations are crystallizations
of physical/natural laws.

Ex) Population growth

Without outside constraints, the growth
rate of a population is directly proportional to
the population size.

$y = \# \text{ of people}$
 $t = \text{time}$

$\frac{dy}{dt} = k \cdot y$

$\frac{dy}{dt}$ is growth rate of population
 k is positive
 y is size of population
D.iff eq for population growth

Notice: if k is positive, then y "function of t " must be
a increasing function
(since $\frac{dy}{dt}$ is positive)

This says the number of babies born is a
constant fraction of the population size

Ex) Newton's Law of Cooling

Describes for example how a cup of coffee cools off.

"Rate of cooling is proportional to the (temperature difference between an object and its surroundings.)"

ambient temperature (constant)

1st step: define variables

I ultimately want a function for temperature of coffee with respect to time.

y = temperature of coffee

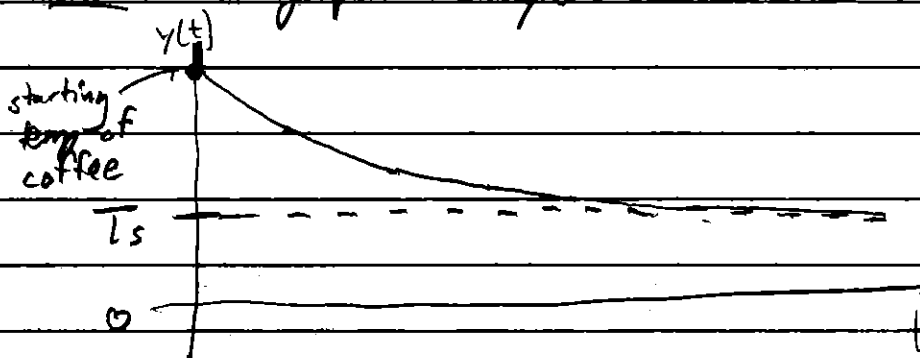
t = time

T_s : temperature of surroundings (constant)

$$(-) \frac{dy}{dt} = k \cdot \underbrace{(y - T_s)}_{T_s - y} ?$$

$$|T_s - y|$$

Guess: for graph of $y(t)$



Population Growth (with constraints)

Here's a better differential equation for population growth: ^{carrying capacity}

y : population size
 t : time

$$\frac{dy}{dt} = k \overset{\text{positive}}{\underset{\text{carrying capacity}}{(3000 - y)}} y$$

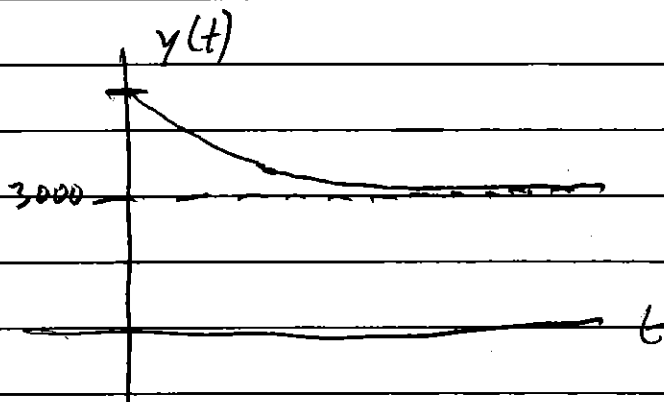
What does $y(t)$ look like?

What if the initial population is > 3000 ?
(ex. 5000)

$$\frac{dy}{dt} = k \underset{\text{negative}}{\overset{\uparrow}{(3000 - y)}} \underset{\text{positive}}{\overset{\uparrow}{y}}$$

If population = 3000,
 $dy/dt = 0$, flat graph
near there

$\frac{dy}{dt}$ will be negative



What if the initial population is 0?

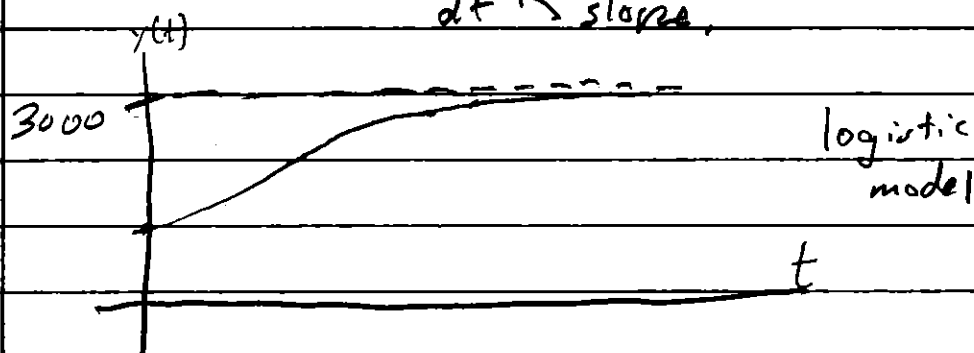
$$\frac{dy}{dt} = \underset{k}{\uparrow} \left(\underset{3000}{\uparrow} - \underset{0}{\uparrow} \right) \underset{0}{\uparrow} \quad \frac{dy}{dt} = 0$$

What if the initial population is 1000?

$$\frac{dy}{dt} = k \left(\underbrace{3000 - y}_{\text{positive}} \right) \underset{\text{positive}}{\uparrow} \quad \frac{dy}{dt} \text{ will be positive whenever } 0 < y < 3000$$

But, when y gets close to 3000,

$3000 - y$ gets close to zero,
so $\frac{dy}{dt}$ also gets close to zero
 $\frac{dy}{dt} \rightarrow \text{slope}$.



If have extra time, solve $y'' + 2y' - 3y = 0$, guessing $y = e^{rt}$, second order differential equation

More complicated analogues, partial differential equations, black scholes equation in econ, maxwell's equations in electrical physics, predator-prey relations, heat equation, etc.